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# SOLVING A SYSTEM OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS BY USING DIAGONALIZATION MATRIX

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**Abstract**

Consider a system of fourth order ordinary differential equation $\boldsymbol{x}^{(4)} = A\boldsymbol{x}$, where $A$ is a square matrix. To find the solution of this system, the matrix $A$ will be diagonalized by using $J = P^{-1}AP$, where $P$ is a matrix with vectoreigens of $A$ as its columns. Therefore, the original system can be transformed into a new system $\boldsymbol{y}^{(4)} = J\boldsymbol{y}$ with $\boldsymbol{y} = P^{-1}\boldsymbol{x}$. Solving this new system will yield the soluton of the original system, that is, $\boldsymbol{x} = P\boldsymbol{y}$. Some problems with different types of eigenvalues of $A$ will be given. Especially, Jordan form is used to diagonalize $A$, when all eigenvalues of $A$ are equal.

Keywords : fourth order ordinary differential equation, diagonalization matrix

## INTRODUCTION

A coupled mass–spring system is governed by a system of second order ordinary differential equations. It is discussed in Nagle [4] and it is solved by assuming the solution is cosine or sine function. An, Yukun [2] discussed a system of second and forth-order ordinary differential equations. In this work, we will study how to solve a system of forth order ordinary differential equations

## BASIC THEORIES

To solve the problem, we need some definitions. The following two definitions are taken from Anton [1].

If $A$ is an $n$ x $n$ matrix, then a nonzero vector $\boldsymbol{x}$ in $R^n$ is called an ***eigenvector*** of $A$ if $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$; that is, if $A\boldsymbol{x} = \boldsymbol{\lambda x}$, for some scalar $\lambda$. The scalar $\lambda$ is called an ***eigenvalue*** of $A$, and $\boldsymbol{x}$ is said to be an eigenvector of $A$ ***corresponding*** to $\lambda$.

A square matrix $A$ is called ***diagonalizable*** if there is an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix; the matrix $P$ is said to ***diagonalize*** $A$.

We now define a special upper-triangular matrix known as a Jordan Block. This definition and the following theorem are taken from Dettman [3]. A Jordan block is a matrix whose elements on the principal diagonal are all equal, whose elements on the first superdiagonal above the principal diagonal are all 1s, and whose elements otherwise are 0s.
Theorem : Jordan Canonical Form
Every square matrix $A$ is similar to an upper-triangular matrix of the form

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & J_m \end{pmatrix}$$

where $J_i$ is a Jordan block with diagonal element $\lambda_i$, an eigenvalue of $A$. An eigenvalue $\lambda_i$ may occur in more than one block, but the number of blocks which contain $\lambda_i$ on the diagonal is equal to the number of independent eigenvectors corresponding to $\lambda_i$.

**RESEARCH METHOD**

In this section, we will show how to solve a system of fourth order homogeneous ordinary differential equations with constant coefficients. We will follow the method developed in Dettman [3] to solve a system of first order differential equations.

Now, consider a system of fourth order homogeneous ordinary differential equations with constant coefficients as follows :

$$\begin{aligned} x_1^{(4)} &= a_1x_1 + a_2x_2 + a_3x_3, \\ x_2^{(4)} &= b_1x_1 + b_2x_2 + b_3x_3, \\ x_3^{(4)} &= c_1x_1 + c_2x_2 + c_3x_3, \end{aligned} \tag{1.1}$$

The system (1.1) can be written as a matrix equation as follows :

$$X^{(4)} = AX, \text{ where} \tag{1.2}$$

$$X^{(4)} = \begin{bmatrix} x_1^{(4)} \\ x_2^{(4)} \\ x_3^{(4)} \end{bmatrix}, \qquad A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

To solve the system (1.1), at first we diagonalize matrix $A$. By finding its eigenvalues and its corresponding eigenvectors, we obtain a matrix $P$ whose columns are eigenvectors of $A$. Therefore, we have

$$P^{-1}AP = J, \tag{1.3}$$

where $J$ is a diagonal matrix, that is, $J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, where $\lambda_i$ are eigenvalues of $A$.

Now, let $Y = P^{-1}X$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. By differentiating both sides, we have

$$Y^{(4)} = P^{-1}X^{(4)}.$$

Substituting Equation (1.2), yields

$$Y^{(4)} = P^{-1}AX,$$

or

$$Y^{(4)} = P^{-1}APP^{-1}X,$$

Using Equation (1.3) and the definition of $Y$ , to obtain

$$Y^{(4)} = JY,$$

or we have a system of forth order differential equations as follows :

$$\begin{aligned} y_1^{(4)} &= \lambda_1 y_1, \\ y_2^{(4)} &= \lambda_2 y_2, \\ y_3^{(4)} &= \lambda_3 y_3, \end{aligned} \tag{1.4}$$

Each equations of system (1.4) is a homogeneous forth order differential equation with constant coefficients. We can solve those equations by using characteristics equation to obtain matrix $Y$. Finally, we can obtain the solution of the original system (1.2.) by using equation $X = PY$.

We will give some examples in the next section to illustrates the above procedures.

## RESULT AND DISCUSSION

In this section we give some examples. We will consider several cases of eigenvalues of matrix $A$. The first one is all eigenvalues of $A$ are different.

Example 1: Case that all the eigenvalues are different.

Solve the following system of differential equations.

$$\begin{aligned} x_1^{(4)} &= -x_1 + 4x_2 - 2x_3, \\ x_2^{(4)} &= -3x_1 + 4x_2, \\ x_3^{(4)} &= -3x_1 + x_2 + 3x_3. \end{aligned} \tag{2.1}$$

Solution : At first, we rewrite the above system into matrix equation as follows :

$$X^{(4)} = AX, \text{ where} \tag{2.2}$$

$$X^{(4)} = \begin{bmatrix} x_1^{(4)} \\ x_2^{(4)} \\ x_3^{(4)} \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Next, we find the eigenvalues of $A$ by solving the characteristic polynom

$$\det(A - \lambda I) = 0,$$

or

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

The roots of the polynom which denote the eigenvalues of $A$ are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$. Substituting each of the eigenvalues into the equation

$$A\boldsymbol{u} = \lambda I\boldsymbol{u},$$

to obtain the corresponding eigenvectors, that is,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{bmatrix}.$$

Now, we form a matrix $P$ whose columns are eigenvectors of $A$ and find its inverse $P^{-1}$. To do so, we have

$$P = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} \\ 1 & 1 & \frac{3}{4} \\ 1 & 1 & 1 \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix}.$$

Next, by multiplying $P^{-1}AP$ we will obtain a diagonal matrix $J$ .

$$P^{-1}AP = \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} \\ 1 & 1 & \frac{3}{4} \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = J.$$

This diagonal matrix is used to solve the following system of equations :

$$Y^{(4)} = JY,$$

or

$$\begin{bmatrix} y_1^{(4)} \\ y_2^{(4)} \\ y_3^{(4)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

or

$$\frac{d^4 y_1}{dt^4} = y_1,$$

$$\frac{d^4 y_2}{dt^4} = 2y_2,$$

$$\frac{d^4 y_3}{dt^4} = 3y_3.$$

Notice that each of the above equations is a forth order homogeneous linear equation with constant coefficients. Solving each equation by using characteristic equation, we get

$$y_1(t) = C_1 e^{t} + C_2 e^{-t} + C_3 \sin(t) + C_4 \cos(t),$$

$$y_2(t) = D_1 e^{\sqrt[4]{2}t} + D_2\, e^{-\sqrt[4]{2}t} + D_3 \sin(\sqrt[4]{2}\, t) + D_4 \cos(\sqrt[4]{2}\, t),$$

$$y_3(t) = E_1 e^{\sqrt[4]{3}t} + E_2\, e^{-\sqrt[4]{3}t} + E_3 \sin(\sqrt[4]{3}\, t) + E_4 \cos(\sqrt[4]{3}\, t),$$

Finally, we can obtain the solution of the original system (2.2.) by using equation $X = PY$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{4} \\ 1 & 1 & \frac{3}{4} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

or

$$x_1(t) = y_1(t) + \frac{2}{3} y_2(t) + \frac{1}{4} y_3(t),$$

$$x_2(t) = y_1(t) + y_2(t) + \frac{3}{4} y_3(t),$$

$$x_3(t) = y_1(t) + y_2(t) + y_3(t).$$

<u>Example 2</u> : Case that there are equal eigenvalues.

Solve the following system of differential equations.

$$\begin{aligned} x_1^{(4)} &= 3x_1 - x_2 + 2x_3, \\ x_2^{(4)} &= -x_1 + 2x_2 - x_3, \qquad\qquad (2.3) \\ x_3^{(4)} &= -x_1 + x_2. \end{aligned}$$

Solution : Similar to Example 1, we rewrite the above system into matrix equation as follows :

$$X^{(4)} = AX, \text{ where} \qquad (2.4)$$

$$X^{(4)} = \begin{bmatrix} x_1^{(4)} \\ x_2^{(4)} \\ x_3^{(4)} \end{bmatrix}, \qquad A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

.

The eigenvalues of $A$ are $\lambda = 1$, $\lambda = 2$, $\lambda = 2$, and their corresponding eigenvectors are respectively

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Since there are only two eigenvectors, at this point, we can not construct a matriks $P$ such that $P^{-1}AP = J$, where $J$ is a diagonal matrix. In this case, instead of considering $J$ as a diagonal matrix, we consider $J$ as a Jordan form of matrix $A$. Therefore, we have the Jordan form of matrix $A$ is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Using this matrix, we can find a matrix $P$ such that $P^{-1}AP = J$, or $AP = PJ$. Let

$$P = \begin{bmatrix} 1 & 1 & a \\ 0 & -1 & b \\ -1 & -1 & c \end{bmatrix}.$$

Substituting matrix $P$ into equation $AP = PJ$, yields a system of linear equations

$$a - b + 2c = 1,$$
$$a + c = 1,$$
$$-a + b - 2c = -1.$$

Solving this system of linear equations to obtain

$$a = 1 - c,$$
$$b = c.$$

If we choose $c = 0$, then $b = 0$, and $a = 1$.

Thus, the matrix $P$ is $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

Now, we can solve the system of equations :

$$Y^{(4)} = JY,$$

or

$$\begin{bmatrix} y_1^{(4)} \\ y_2^{(4)} \\ y_3^{(4)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

or

$$\frac{d^4 y_1}{dt^4} = y_1,$$
$$\frac{d^4 y_2}{dt^4} = 2y_2 + y_3,$$
$$\frac{d^4 y_3}{dt^4} = 2y_3.$$

At first, we solve the first differential equation and the third one to obtain

$$y_1(t) = C_1e^t + C_2e^{-t} + C_3 \sin(t) + C_4 \cos(t),$$

and

$$y_3(t) = E_1 e^{\sqrt[4]{2}t} + E_2 e^{-\sqrt[4]{2}t} + E_3 \sin(\sqrt[4]{2}\ t) + E_4 \cos(\sqrt[4]{2}\ t).$$

Notice that the second differential equation is a non homogeneous one. So, it is necessary to find its complementary solution and particular solution by using the method of undetermined equations. Hence,

$$y_2(t) = D_1 e^{\sqrt[4]{2}t} + D_2 e^{-\sqrt[4]{2}t} + D_3 \sin(\sqrt[4]{2}\ t) + D_4 \cos(\sqrt[4]{2}\ t) +$$
$$\frac{E_1}{4\sqrt[4]{8}} t e^{\sqrt[4]{2}t} - \frac{E_2}{4\sqrt[4]{8}} t e^{-\sqrt[4]{2}t} - \frac{E_4}{4\sqrt[4]{8}} t \sin(\sqrt[4]{2}\ t) + \frac{E_3}{4\sqrt[4]{8}} t \cos(\sqrt[4]{2}\ t).$$

Finally, we can obtain the solution of the original system (2.3) by using equation $X = PY$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

or

$$x_1(t) = y_1(t) + y_2(t) + y_3(t),$$
$$x_2(t) = -y_2(t),$$
$$x_3(t) = -y_1(t) - y_2(t).$$

<u>Example 3 :</u> Case that all the eigenvalues are equal.
Solve the following system of differential equations.

$$\begin{aligned} x_1^{(4)} &= 5x_1 + x_3, \\ x_2^{(4)} &= x_1 + x_2, \qquad\qquad (2.5) \\ x_3^{(4)} &= -7x_1 + x_2. \end{aligned}$$

Solution : Similar to Example 1, we rewrite the above system into matrix equation as follows :

$$X^{(4)} = AX, \text{ where} \qquad (2.6)$$

$$X^{(4)} = \begin{bmatrix} x_1^{(4)} \\ x_2^{(4)} \\ x_3^{(4)} \end{bmatrix}, \qquad A = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The eigenvalues of $A$ are $\lambda = 2$, $\lambda = 2$, $\lambda = 2$, and their corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}.$$

Since there is only one corresponding eigenvector, we have the Jordan form of matrix $A$ is

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Letting

$$P = \begin{bmatrix} 1 & a & d \\ 1 & b & e \\ -3 & c & f \end{bmatrix}.$$

and using equation $AP = PJ$, we will have a system of linear equations
Solving this system of linear equations to obtain

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -3 & 1 & 0 \end{bmatrix}, \text{ and } P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Now, we can solve the system of equations :

$$Y^{(4)} = JY,$$

or

$$\begin{bmatrix} y_1^{(4)} \\ y_2^{(4)} \\ y_3^{(4)} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

or

$$\frac{d^4 y_1}{dt^4} = 2y_1 + y_2,$$

$$\frac{d^4 y_2}{dt^4} = 2y_2 + y_3,$$

$$\frac{d^4 y_3}{dt^4} = 2y_3.$$

At first, we solve the third differential equation to obtain

$$y_3(t) = E_1 e^{\sqrt[4]{2}t} + E_2\, e^{-\sqrt[4]{2}t} + E_3 \sin(\sqrt[4]{2}\,t) + E_4\cos(\sqrt[4]{2}\,t).$$

Next, we solve the second differential equation that is a non homogeneous equation. Using the method of undetermined equations, yields

$$y_2(t) = D_1 e^{\sqrt[4]{2}t} + D_2\, e^{-\sqrt[4]{2}t} + D_3 \sin(\sqrt[4]{2}\,t) + D_4\cos(\sqrt[4]{2}\,t) + \frac{E_1}{4\sqrt[4]{8}}\, t\, e^{\sqrt[4]{2}t} - \frac{E_2}{4\sqrt[4]{8}}\, t e^{-\sqrt[4]{2}t} - \frac{E_4}{4\sqrt[4]{8}}\, t \sin(\sqrt[4]{2}\,t) + \frac{E_3}{4\sqrt[4]{8}}\, t \cos(\sqrt[4]{2}\,t).$$

Finally, we have the solution of the first equation

$$y_1(t) = C_1 e^{\sqrt[4]{2}t} + C_2 e^{-\sqrt[4]{2}t} + C_3 \sin(\sqrt[4]{2}\ t) + C_4\cos(\sqrt[4]{2}\ t) +$$

$$\left(\frac{16D_1 - 3E_1}{64\sqrt[4]{8}}\right) t e^{\sqrt[4]{2}t} + \frac{E_1\sqrt{2}}{128} t^2 e^{\sqrt[4]{2}t} +$$

$$\left(\frac{-16D_2 + 3E_2}{64\sqrt[4]{8}}\right) t e^{-\sqrt[4]{2}t} + \frac{E_2\sqrt{2}}{128} t^2 e^{-\sqrt[4]{2}t} +$$

$$\left(\frac{-16D_4 + 3E_4}{64\sqrt[4]{8}}\right) t \sin(\sqrt[4]{2}\ t) - \frac{E_3\sqrt{2}}{128} t^2 \sin(\sqrt[4]{2}\ t) +$$

$$\left(\frac{16D_3 - 3E_3}{64\sqrt[4]{8}}\right) t \cos(\sqrt[4]{2}\ t) - \frac{E_4\sqrt{2}}{128} t^2 \cos(\sqrt[4]{2}\ t).$$

Finally, we can obtain the solution of the original system (2.5) by using equation $X = PY$,

$$x_1(t) = y_1(t),$$
$$x_2(t) = y_1(t) - y_2(t) + y_3(t),,$$
$$x_3(t) = -3y_1(t) + y_2(t).$$

## CONCLUSION AND SUGGESTION

We have discussed a method to solve a system of fourth order homogeneous ordinary differential equations with constant coefficients

$$X^{(4)} = AX,$$

by diagonalizing matrix $A$, or considering the Jordan form of matrix $A$.

In the future research it will be discussed the nonhomogeneous system and a system of the form

$$X^{(4)} + X^{(2)} = AX.$$

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